Monotone measures–based integrals: special functionals and an optimization tool

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Considering the space of all non–negative measurable functions $\mathcal{F}_{(X,\mathcal{A})}$ linked to a fixed measurable space $(X,\mathcal{A})$, the Lebesgue integral can be seen as an additive, continuous from below functional $\mathcal{L}$ on $\mathcal{F}_{(X,\mathcal{A})}$, related to a measure $m : \mathcal{A} \to [0,\infty]$ given by $m(A) = \mathcal{L}(1_A)$. We introduce several other integrals which can be seen as special functionals on $\mathcal{F}_{(X,\mathcal{A})}$. For example, the Choquet integral [7] is a comonotone additive functional $\mathcal{C}$, and the corresponding monotone measure $m : \mathcal{A} \to [0,\infty]$ is given by $m(A) = \mathcal{C}(1_A)$. Note that though the first traces of Choquet integral goes back to Vitali [18], its comonotone additivity was stressed by [15, 16] based on inputs from economy. Similarly, the Sugeno integral $\mathcal{S}u$ [17] is a min–homogeneous comonotone maxitive functional on $\mathcal{F}_{(X,\mathcal{A})}$. Among several approaches to integration with respect to monotone measures, covering both the Choquet and the Sugeno integrals, we recall the axiomatic approach proposed by Benvenuti et al. [1]. A rather general framework of universal integrals was recently proposed in [8], unifying the look on both Choquet and Sugeno integral. Two alternative looks on universal integrals are offered in our paper [5].

Some integrals were introduced as special functionals. We recall the concave integral Cov introduced by Lehrer [11], which, for a given monotone measure m on $\mathcal{A}$, is the smallest positively homogeneous concave functional on $\mathcal{F}_{(X,\mathcal{A})}$ such that $Cov(1_A) \geq m(A)$. Similarly, the convex integral Con was introduced in [13].

On the other hand, several new types of integrals were or can be introduced as optimization tools. For a given system $\mathcal{H}$ of set systems from $\mathcal{A}$, Even and Lehrer [2] have proposed the decomposition integral $\mathcal{D}_\mathcal{H} : \mathcal{F}_{(X,\mathcal{A})} \to [0,\infty]$, which, for a given monotone measure m on $\mathcal{A}$, is given by

$$\mathcal{D}_\mathcal{H}(f) = \sup \left\{ \sum_{i\in I} a_i\, m(A_i) \,|\, a_i \geq 0, (A_i)_{i\in I} \in \mathcal{H}, \sum_{i\in I} a_i\, 1_{A_i} \leq f \right\}.$$

Obviously, if $\mathcal{H}$ consists of finite (countable) partitions of $(X,\mathcal{A})$ and m is a measure, then $\mathcal{D}_\mathcal{H} = \mathcal{L}$ (if m is a monotone measure, then $\mathcal{D}_\mathcal{H}$ is the PAN–integral [19]). If $\mathcal{H}$ consists of all finite (countable) chains, then $\mathcal{D}_\mathcal{H} = \mathcal{C}$. If $\mathcal{H}$

consists of all singleton systems, $\mathcal{H} = \{\{A\} \mid A \in \mathcal{A}\}$, $\mathcal{D}_{\mathcal{H}} = Sh$ is the Shilkret integral. If $\mathcal{H}$ consists of all finite subsystems of $\mathcal{A}$, then $\mathcal{D}_{\mathcal{H}} = Cov$.

An alternative look, based on superdecomposition, was introduced in [13], yielding the superdecomposition integral $\mathcal{D}^{\mathcal{H}}$ (and defined for bounded functions f only)

$$\mathcal{D}^{\mathcal{H}}(f) = \inf \left\{ \sum_{i \in I} a_i \, m(A_i) \mid a_i \geq 0, (A_i)_{i \in I} \in \mathcal{H}, \sum_{i \in I} a_i \, 1_{A_i} \geq f \right\}.$$

We will introduce a distinguished family of universal integrals based on copulas and related to decomposition and superdecomposition integrals (for more details see [10]). We will stress the integrals based on the two distinguished copulas. Namely, the independence copula Π (product) yields the Choquet integral, while the comonotonicity copula M (min) yields the Sugeno integral.

Finally, we will indicate the latest streamings in the area, especially the concept of superadditive and subadditive integrals based on some aggregation functions [6]. We will touch also the concept of bipolar integrals, especially the bipolar universal integrals [4]. Promising is also the concept of level–dependent capacities and related integrals [3, 12, 9].

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